An L^p-theory of non-divergence form SPDEs driven by Lévy processes

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Abstract

In this paper we present an L^p-theory for the stochastic partial differential equations (SPDEs in abbreciation) driven by Lévy processes. Existence and uniqueness of solutions in Sobolev spaces are obtained. The coefficients of SPDEs under consideration are random functions depending on time and space variables.

Keywords: Stochastic partial differential equation, Lévy process, L^p-theory, Sobolve space, martingale.

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, P)$-null sets. We assume that on Ω we are given independent one-dimensional Lévy processes $Z^1_t, Z^2_t, ...$ relative to $\{\mathcal{F}_t, t \geq 0\}$. Let $\mathcal{P}$ be the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$.

In this article we are dealing with $W^{n,p}$-theory of the stochastic partial differential equation

$$du = (a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f)dt + (\sigma^{ik}u_{x^i} + \mu^k u + g^k)dZ^k_t \tag{1.1}$$

given for $t \geq 0$ and $x \in \mathbb{R}^d$. Here $p \in [2, \infty)$ and $n \in \mathbb{R}$. Indices i and j go from 1 to d, and k runs through $\{1, 2, ...\}$ with the summation convention on i, j, k being enforced. The coefficients a^{ij}, b^i, c, σ^{ik}, μ^k and the free terms f, g^k are random functions depending on (t, x).

Demand for a general theory of stochastic partial differential equations (SPDEs) driven by Lévy processes is obvious when we model a natural phenomenon with randomness and jumps. The main objective of this paper is to establish unique solvability in Sobolev spaces for SPDEs (1.1).

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If $\{Z^k, k \geq 1\}$ are independent one-dimensional Wiener processes, L^p-theory for SPDE (1.1) has been well studied. An L^p-theory of SPDEs with Wiener processes defined on $\mathbb{R}^n$ was first introduced by Krylov in [7]. Subsequently, Krylov and Lototsky [9, 10] developed an L^p-theory of such equations in half space $\mathbb{R}^n_+$ with constant coefficients. These results were later extended to SPDEs with variable coefficients defined in bounded domains of $\mathbb{R}^n$ by several authors, see, for instance, [6, 5, 11].

However very little is known when $\{Z^k, k \geq 1\}$ are general discontinuous Lévy processes. As far as we know, most previous work on SPDEs driven by Lévy processes deal with equations with non-random coefficients independent of t and, moreover, σ^{ik} have always been assumed to be zero, consequently first derivatives of solutions were not allowed to appear in the stochastic part. More precisely, the typical type of equations appearing in the previous works (see [1, 4, 12, 13] and references therein) is of the following type:

$$du = (Au + f)dt + \sum_{k=1}^{n} g^k(u) dZ^k_t, \tag{1.2}$$

where $A(t)$ is the generator of certain semigroup. and the function g^k satisfies certain continuity conditions.

Our approaches are different from those in [1, 4, 12, 13]. We adopt analytic approaches introduced by Krylov in [7]. Our results are new even for the stochastic heat equation

$$du = \Delta u \, dt + g \, dZ_t,$$

since we establish unique solvability result in $L^p(\Omega \times [0, T], H^n_p)$ for every $p \in [2, \infty)$ and $n \in \mathbb{R}$, not just in $L^2(\Omega \times [0, T], H^1_2)$ space. (The definition of Sobolev space $H^n_p = W^{n,p}$ will be given in next section.) This allows us to obtain various regularity results of solutions. See Remark 2.10(ii).

Our main results are stated in section 2 and consist of Theorem 2.7 (L^2-theory) and Theorem 2.9 (L^p-theory, $p > 2$). In section 3 we deal with equations with constant coefficients, and in section 4 we prove Theorem 2.7 and Theorem 2.9.

We end the introduction with some notation. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$, and $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$. For $i = 1, ..., d$, multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_i \in \{0, 1, 2, ...\}$ and functions $u(x)$ on $\mathbb{R}^d$, we set

$$D_i u := u_{x^i} := \partial u / \partial x^i, \qquad D^\alpha u := D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d} u, \qquad |\alpha| := \alpha_1 + ... + \alpha_d.$$

For $a, b \in \mathbb{R}^d$, we define $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For $p \geq 1$, we will use $\|u\|_p$ to denote the L^p-norm of u in $L^p(\mathbb{R}^d; dx)$. For scalar functions f, g on $\mathbb{R}^d$, $(f, g) := \int_{\mathbb{R}^d} f(x)g(x)dx$.

2 Main results

For $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, define

$$N^k(t, A) := \# \left\{ 0 \leq s \leq t; \, Z^k_s - Z^k_{s-} \in A \right\}, \quad \widetilde{N}^k(t, A) := N^k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1,A)]$ is the Lévy measure of Z^k. By Lévy-Itô decomposition, there exist constants α^k, β^k and Brownian motion B^k so that

$$Z^k(t) = \alpha^k t + \beta^k B^k_t + \int_{|z|<1} z\widetilde{N}^k(t,dz) + \int_{|z|\geq 1} zN^k(t,dz). \tag{2.1}$$

For simplicity, throughout this paper, we assume $\alpha^k = \beta^k = 0$. Consider the following equation for random function $u(t,x)$ on $\Omega \times [0,\infty) \times \mathbb{R}^d$:

$$du = \left(a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f\right)dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k\right)dZ^k_t \tag{2.2}$$

in the weak sense. Precise definition of weak solution to (2.2) will be given in Definition 2.4 below. Here i and j go from 1 to d, and k runs through $\{1,2,\cdots\}$. The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \mu^k$ and the free terms f, g^k are random functions depending on $t>0$ and $x \in \mathbb{R}^d$.

Assumption 2.1 *$p \in [2,\infty)$ and for each k,*

$$\widehat{c}_{k,p} := \left(\int_{\mathbb{R}} |z|^p \nu_k(dz)\right)^{1/p} < \infty.$$

For $n = 0,1,2,...$, define Sobolev space

$$H^n_p := H^n_p(\mathbb{R}^d) = \left\{u : u, Du, ..., D^n u \in L^p(\mathbb{R}^d)\right\},$$

where $D^k u$ are derivatives in the distributional sense. In literature, H^n_p is also denoted as $W^{n,p}(\mathbb{R}^d)$. In general, for $\gamma \in \mathbb{R}$ define the space $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2}L^p$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all distributions u such that $(1-\Delta)^{\gamma/2}u \in L^p$. For $u \in H^\gamma_p$, we define

$$\|u\|_{H^\gamma_p} := \|(1-\Delta)^{\gamma/2}u\|_p := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\gamma/2}\mathcal{F}(u)(\xi)]\|_p, \tag{2.3}$$

where $\mathcal{F}$ is the Fourier transform. Let $\mathcal{P}^{dP\times dt}$ be the completion of $\mathcal{P}$ with respect to $dP \times dt$. Denote by $\mathbb{H}^\gamma_p(T)$ the space of all $\mathcal{P}^{dP\times dt}$-measurable processes $u : [0,T]\times\Omega \to H^\gamma_p$ so that

$$\|u\|_{\mathbb{H}^\gamma_p(T)} := \left(\mathbb{E}\left[\int_0^T \|u\|^p_{H^\gamma_p}\,dt\right]\right)^{1/p} < \infty.$$

For fixed $p \geq 2$, define

$$\widehat{c}_k := \widehat{c}_{k,2} \vee \widehat{c}_{k,p}.$$

Note that $\widehat{c}_k = \widetilde{c}_{k,2}$ when $p=2$, and for $2<q<p$, by Hölder's inequality,

$$\widehat{c}_{k,q} \leq \left(\int_{\mathbb{R}} |z|^2\nu_k(dz)\right)^{(p-q)/(q(p-2))} \left(\int_{\mathbb{R}} |z|^p\nu_k(dz)\right)^{(q-2)/(q(p-2))} \leq \widehat{c}_k.$$

For ℓ^2-valued processes $g = (g^1, g^2, ...)$, we say $g \in \mathbb{H}^\gamma_p(T,\ell^2)$ if $g^k \in \mathbb{H}^\gamma_p(T)$ for every $k \geq 1$ and

$$\|g\|_{\mathbb{H}^\gamma_p(T,\ell^2)} := \left(\mathbb{E}\left[\int_0^T \|\,|(1-\Delta)^{\gamma/2}\widehat{g}|_{\ell^2}\,\|^p_p\,dt\right]\right)^{1/p} < \infty, \tag{2.4}$$

where $\widehat{g} = (\widehat{g}_1, \widehat{g}_2, \widehat{g}_3, \cdots) := (\widehat{c}_1 g^1, \widehat{c}_2 g^2, \widehat{c}_3 g^3, \cdots)$. Finally, we say $u_0 \in U_p^\gamma$ if u_0 is $\mathcal{F}_0$-measurable and

$$\|u_0\|_{U_p^\gamma} := \left(\mathbb{E}\left[\|u_0\|^p_{H_p^{\gamma-(2/p)}}\right]\right)^{1/p} < \infty.$$

Remark 2.2 It follows from (2.3) that for any $\mu, \gamma \in \mathbb{R}$, the operator $(1-\Delta)^{\mu/2} : H_p^\gamma \to H_p^{\gamma-\mu}$ is an isometry. Indeed,

$$\|(1-\Delta)^{\mu/2} u\|_{H_p^{\gamma-\mu}} = \|(1-\Delta)^{(\gamma-\mu)/2}(1-\Delta)^{\mu/2} u\|_p = \|(1-\Delta)^{\gamma/2} u\|_p = \|u\|_{H_p^\gamma}.$$

Remark 2.3 (i) Let $\mathcal{M}_p^\gamma(T)$ denote the set of all H_p^γ-valued $\{\mathcal{F}_t\}$-adapted processes $u(t)$ that are $\mathcal{F} \otimes \mathcal{B}(0,T)$-measurable and satisfy

$$\mathbb{E}\left[\int_0^T \|u\|^p_{H_p^\gamma}\, dt\right] < \infty.$$

Then by Theorem 2.8.2 in [8], $\mathcal{M}_p^\gamma(T) \subset \mathbb{H}_p^\gamma(T)$.

(ii) Note that under Assumption 2.1 for $p = 2$, Z^k is a square integrable martingale for each $k \geq 1$. For every $\mathcal{P}^{dP\times dt}$-measurable process $H \in L^2(\Omega \times [0,T])$, $M_t := \int_0^t H_s dZ_s^k$ is a square integrable martingale with

$$\mathbb{E}[M_t^2] = \mathbb{E}\left[\int_0^t H_s^2\, d[Z^k]_s\right] = \widehat{c}^2_{k,2}\mathbb{E}\left[\int_0^t H_s^2 ds\right]. \tag{2.5}$$

So for any $g \in \mathbb{H}_p^\gamma(T, \ell^2)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\sum_{k=1}^\infty \widehat{c}_k^2\, \mathbb{E}\left[\int_0^T (g^k, \phi)^2 ds\right] \\
= \;& \sum_{k=1}^\infty \mathbb{E}\left[\int_0^T ((1-\Delta)^{\gamma/2}\widehat{c}_k g^k, (1-\Delta)^{-\gamma/2}\phi)^2\, ds\right] \\
\leq \;& \|(1-\Delta)^{-\gamma/2}\phi\|_1\, \mathbb{E}\left[\int_0^T \left(\sum_k |(1-\Delta)^{\gamma/2}\widehat{c}_k g^k|^2,\, |(1-\Delta)^{-\gamma/2}\phi|\right) ds\right] \\
\leq \;& \|(1-\Delta)^{-\gamma/2}\phi\|_1\, \|(1-\Delta)^{-\gamma/2}\phi\|_q\, \mathbb{E}\left[\int_0^T \left\| \sum_k |(1-\Delta)^{\gamma/2}\widehat{c}_k g^k|^2 \right\|_{p/2} ds\right] \\
\leq \;& \|(1-\Delta)^{-\gamma/2}\phi\|_1\, \|(1-\Delta)^{-\gamma/2}\phi\|_q\, T^{1-\frac{2}{p}}\, \|g\|^2_{\mathbb{H}_p^\gamma(T,l^2)} < \infty,
\end{aligned}$$

where $q = p/(p-2)$. Thus in view of (2.5), the series of stochastic integral $\sum_{k=1}^\infty \int_0^t (g^k, \phi) dZ_s^k$ defines a square integrable martingale on $[0,T]$, which is right continuous with left limits.

Definition 2.4 *Write $u \in \mathcal{H}_p^{\gamma+2}(T)$ if $u \in \mathbb{H}_p^{\gamma+2}(T)$ with $u(0) \in U_p^\gamma$, and for some $f \in \mathbb{H}_p^\gamma(T)$ and $g \in \mathbb{H}_p^{\gamma+1}(T, \ell^2)$*

$$du = f dt + g^k dZ_t^k, \quad \textit{for } t \in [0,T]$$

in the distributional sense, that is, for any $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$(u(t),\phi) = (u(0),\phi) + \int_0^t (f,\phi)dt + \sum_k \int_0^t (g^k,\phi)dZ_t^k \tag{2.6}$$

holds for all $t \le T$ *a.s.. Define*

$$\mathbb{D}u := f, \quad \mathbb{S}u := g,$$

and define

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} := \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|\mathbb{D}u\|_{\mathbb{H}_p^{\gamma}(T)} + \|\mathbb{S}u\|_{\mathbb{H}_p^{\gamma+1}(T,\ell^2)} + \|u(0)\|_{U_p^{\gamma+2}}.$$

Theorem 2.5 *For any* $p \in [2,\infty)$,$\gamma \in \mathbb{R}$ *and* $T > 0$, $\mathcal{H}_p^{\gamma+2}(T)$ *is a Banach space with norm* $\|\cdot\|_{\mathcal{H}_p^{\gamma+2}(T)}$. *Moreover, there is a constant* $c = c(d,p) > 0$, *independent of* T, *such that for every* $u \in \mathcal{H}_p^{\gamma+2}(T)$,

$$\mathbb{E}\left[\sup_{t\le T}\|u(t)\|_{H_p^\gamma}^p\right] \le c\left(\|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(T)}^p + \|\mathbb{S}u\|_{\mathbb{H}_p^\gamma(T,\ell^2)}^p + \mathbb{E}\left[\|u_0\|_{H_p^\gamma}^p\right]\right). \tag{2.7}$$

Consequently, for each $t > 0$,

$$\|u\|_{\mathbb{H}_p^\gamma(t)}^p \le c\int_0^t \|u\|_{\mathcal{H}_p^{\gamma+2}(s)}^p ds. \tag{2.8}$$

Proof. In view of Remark 2.2 it suffices to prove the theorem for $\gamma = 0$. First we prove (2.7). Let $du = fdt + g^k dZ_t^k$ with $u(0) = u_0$. Assume that $g^k = 0$ for all $k \ge N_0$ and g^k is of the type

$$g^k(t,x) = \sum_{i=0}^m I_{(\tau_i^k,\tau_{i+1}^k]}(t)g^{ki}(x), \tag{2.9}$$

where τ_i^k are bounded stopping times and $g^{ki} \in C_c^\infty(\mathbb{R}^d)$. Define

$$v(t,x) = \sum_{k=1}^\infty \int_0^t g^k dZ_s^k.$$

Then by Burkholder-Davis-Gundy inequality (used twice) and monotone convergence theorem,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\le t}|v(s,x)|^p\right]\\
\le\ & c\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int |g^k(s,x)|^2|z|^2N^k(ds,dz)\right)^{p/2}\right]\\
=\ & c\lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^2|z|^2N^k(ds,dz)\right)^{p/2}\right]\\
\le\ & c\lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^2|z|^2\widetilde{N}^k(ds,dz)\right)^{p/2}\right]\\
&+c\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{\mathbb{R}}\sum_{k=1}^{\infty}|g^k(s,x)|^2\,|z|^2\nu_k(dz)ds\right)^{p/2}\right]\\
\le\ & c\lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^4|z|^4N^k(ds,dz)\right)^{p/4}\right]+c\mathbb{E}\left[\left(\int_0^t\sum_{k=1}^{\infty}|\widehat{g}^k(s,x)|^2\,ds\right)^{p/2}\right].
\end{aligned}
$$

Recall that for any $q>1$, $(\sum|a_n|^q)^{1/q}\le\sum|a_n|$. Thus if $2<p\le 4$, then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^4\,|z|^4N^k(ds,dz)\right)^{p/4}\right]\\
\le\ & \mathbb{E}\left[\left(\sum_k\sum_{0\le s\le t}|g^k(s,x)|^4\,|\Delta Z_s^k|^4\right)^{p/4}\right]\le\mathbb{E}\left[\sum_k\sum_{0\le s\le t}|g^k(s,x)|^p\,|\Delta Z_s^k|^p\right]\\
=\ & \mathbb{E}\left[\int_0^t\sum_{k=1}^{\infty}|\widehat{c}_{k,p}g^k(s,x)|^p\,ds\right]\le\mathbb{E}\left[\int_0^t\left(\sum_{k=1}^{\infty}|\widehat{c}_{k,p}g^k(s,x)|^2\right)^{p/2}ds\right]
\end{aligned}
$$

If $4<p\le 8$ then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^4\,|z|^4N^k(ds,dz)\right)^{p/4}\right]\\
\le\ & \mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^4\,|z|^4\widetilde{N}^k(ds,dz)+\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N}|g(s,x)|^4\,|z|^4\nu_k(dz)ds\right)^{p/4}\right]\\
\le\ & c\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t\int_{|z|\le N} |g^k(s,x)|^8|z|^8N^k(ds,dz)\right)^{p/8}+\left(\int_0^t\sum_{k=1}^{\infty}|\widehat{g}^k(s,x)|^4ds\right)^{p/4}\right]\\
\le\ & c\mathbb{E}\left[\int_0^t\sum_{k=1}^{\infty}|\widehat{c}_{k,p}g^k(s,x)|^p\,ds+\left(\int_0^t\sum_{k=1}^{\infty}|\widehat{g}^k(s,x)|^4ds\right)^{p/4}\right].
\end{aligned}
$$

Similarly, in general, for $p \in (2^{n-1}, 2^n]$,

$$
\begin{aligned}
& \mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t \int |g^k(s,x)|^2\,|z|^2 N^k(dz,ds)\right)^{p/2}\right] \\
\leq\ & c\,\mathbb{E}\left[\sum_{j=1}^{n}\left(\int_0^t \sum_k |\widehat{g}^k(s,x)|^{2j} ds\right)^{p2^{-j}}\right] + c\,\mathbb{E}\left[\int_0^t \sum_k |\widehat{g}^k(s,x)|^p ds\right].
\end{aligned}
$$

Also since for each $2 \leq q \leq p$,

$$
\left(\int_0^t \sum_{k=1}^{\infty} |\widehat{g}^k(s,x)|^q ds\right)^{1/q} \leq c(q)\left(\left(\int_0^t \sum_{k=1}^{\infty} |\widehat{g}^k(s,x)|^2 ds\right)^{1/2} + \left(\int_0^t \sum_{k=1}^{\infty} |\widehat{g}^k(s,x)|^p ds\right)^{1/p}\right),
$$

we get

$$
\begin{aligned}
& \mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t \int |g^k(s,x)|^2\,|z|^2 N^k(dz,ds)\right)^{p/2}\right] \\
\leq\ & c(p)\,\mathbb{E}\left[\left(\int_0^t \sum_k |\widehat{g}^k(s,x)|^2 ds\right)^{p/2} + \int_0^t \sum_k |\widehat{g}^k(s,x)|^p\, ds\right]
\end{aligned}
\tag{2.10}
$$

and

$$
\mathbb{E}\left[\sup_{s\leq t} |v(s,x)|^p\right] \leq c(p)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{\infty} |\widehat{g}^k(s,x)|^2 ds\right)^{p/2} + \int_0^t \sum_k |\widehat{g}^k(s,x)|^p ds\right]. \tag{2.11}
$$

By integrating over $\mathbb{R}^d$, we get

$$
\mathbb{E}\left[\sup_{s\leq t} \|v\|_p^p\right] \leq c(p)\|g\|^p_{\mathbb{H}^0_p(t,\ell^2)}. \tag{2.12}
$$

Next we show that (2.12) holds for general $g \in \mathbb{H}^0_p(T,\ell^2)$. Take a sequence $g_n \in \mathbb{H}^0_p(T,\ell^2)$ so that for each fixed n, $g_n^k = 0$ for all large k and each g_n^k is of of the type (2.9), and $g_n \to g$ in $\mathbb{H}^0_p(T,\ell^2)$ as $n \to \infty$. Define $v_n(t,x) = \sum_k \int_0^t g_n^k dZ_t^k$, then

$$
\mathbb{E}\left[\sup_{s\leq t} \|v_n\|_p^p\right] \leq c(p)\|g_n\|^p_{\mathbb{H}^0_p(t,\ell^2)}, \quad \mathbb{E}\left[\sup_{s\leq t} \|v_m - v_n\|_p^p\right] \leq c(p)\|g_m - g_n\|^p_{\mathbb{H}^0_p(t,\ell^2)}.
$$

Thus (2.12) follows by taking $n \to \infty$. Now note that

$$
d(u-v) = f dt \quad \text{with} \quad (u-v)(0) = u_0.
$$

Thus it is easy to check that

$$
\mathbb{E}\left[\sup_{s\leq t} \|u-v\|_p^p\right] \leq c\mathbb{E}\left[\|u_0\|_p^p\right] + c\mathbb{E}\left[\int_0^t \|f(s,\cdot)\|_p^p\, ds\right].
$$

Consequently,

$$\mathbb{E}\left[\sup_{s\le t}\|u\|_p^p\right] \le c\|f\|^p_{\mathbb{H}^0_p(t)} + c\|g\|^p_{\mathbb{H}^0_p(t,\ell^2)} + c\mathbb{E}\|u_0\|^p_{L_p}.$$

The completeness of the space $\mathcal{H}^2_p(T)$ easily follows from (2.7). The theorem is proved. □

Now we introduce the space of point-wise multipliers in H^γ_p. Fix $\kappa_0 > 0$. For $r \ge 0$, define $r_+ = r$ if $r = 0, 1, 2, \cdots$, and $r_+ = r + \kappa_0$ otherwise. Also denote $r^+ = r + \kappa_0$. Define

$$B^r = \begin{cases} B(\mathbb{R}^d) & \text{if } r = 0, \\ C^{r-1,1}(\mathbb{R}^d) & \text{if } r = 1, 2, \cdots, \\ C^r(\mathbb{R}^d) & \text{otherwise}, \end{cases} \tag{2.13}$$

where $B(\mathbb{R}^d)$ is the space of bounded Borel measurable functions on $\mathbb{R}^d$, $C^{r-1,1}(\mathbb{R}^d)$ is the space of $r-1$ times continuously differentiable functions whose $(r-1)$st order derivatives are Lipschitz continuous, and $C^r(\mathbb{R}^d)$ is the usual Hölder space. We also use the Banach space B^r for ℓ^2-valued functions. For instance, if $g = (g^1, g^2, ...)$, then $|g|_{B^0} = \sup_x |g(x)|_{\ell^2}$ and

$$|g|_{C^{n-1,1}} = \sum_{|\alpha|\le n-1} |D^\alpha g|_{B^0} + \sum_{|\alpha|=n-1} \sup_{x\ne y} \frac{|D^\alpha g(x) - D^\alpha g(y)|_{\ell^2}}{|x-y|}.$$

Assumption 2.6 *(i) The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \mu^k$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions.*

(ii) $a^{ij} = a^{ji}$, and the functions a^{ij} and σ^i are uniformly continuous in x. In other words, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-y| < \delta$,

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell^2} < \varepsilon.$$

(iii) There exist constants $\delta, K > 0$ so that

$$|a^{ij}| + |b^i| + |c| + |\sigma^i|_{\ell^2} + |\mu|_{\ell^2} \le K,$$

$$\delta I_{d\times d} \le (a^{ij} - \alpha^{ij}) \le (a^{ij}) \le K I_{d\times d}, \tag{2.14}$$

where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^\infty \hat{c}^2_{k,2}\sigma^{ik}\sigma^{jk}$ and $I_{d\times d}$ denotes the $(d\times d)$-identity matrix.

Here are main results of this article. We formulate them into two theorems since our assumptions are stronger when $p \ne 2$.

Theorem 2.7 *Let $\gamma \in \mathbb{R}, T > 0$ and Assumption 2.6 hold. Also assume there is a constant $L > 0$ so that for each ω, t,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\sigma^i(t,\cdot)|_{B^{|\gamma+1|_+}} + |\mu(t,\cdot)|_{B^{|\gamma+1|_+}} \le L.$$

Then for any $f \in \mathbb{H}^\gamma_2(T)$, $g \in \mathbb{H}^{\gamma+1}_2(T,\ell^2)$ and $u_0 \in U^{\gamma+2}_2$ equation (2.2) has a unique solution $u \in \mathcal{H}^{\gamma+2}_2(T)$, and

$$\|u\|_{\mathcal{H}^{\gamma+2}_2(T)} \le c\left(\|f\|_{\mathbb{H}^\gamma_2(T)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}\right), \tag{2.15}$$

where $c = c(\delta, K, L, \gamma, T)$.

Remark 2.8 *Condition (2.14) naturally appears when one writes Itô's formula for* $|u|^2$*, where* u *is a solution of (1.1) (see Lemma 2.8 in [3]). Remember we assumed* $\beta^k = 0$ *in (2.1). If* $\beta^k \neq 0$ *we need to replace (2.14) by*

$$\delta I_{d\times d} \leq (a^{ij} - \bar{\alpha}^{ij}) \leq (a^{ij}) \leq K I_{d\times d},$$

where $\bar{\alpha}^{ij} := \frac{1}{2}\sum_{k=1}^{\infty}((\beta^k)^2 + \widehat{c}^2_{k,2})\sigma^{ik}\sigma^{jk}$*.*

Theorem 2.9 *Let* $p \in (2,\infty), \gamma \in \mathbb{R}$ *and* $\varepsilon > 0$ *be fixed. Assume Assumption 2.6 holds,* $\sigma^{ik} = 0$ *and there is a constant* $L > 0$ *so that for each* ω, t*,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\mu(t,\cdot)|_{B^{|\gamma+1|^+}} \leq L.$$

Then for any $f \in \mathbb{H}^\gamma_p(T)$*,* $g \in \mathbb{H}^{\gamma+1+\varepsilon}_p(T,\ell^2)$ *and* $u_0 \in U^{\gamma+2}_p$*, equation (2.2) has a unique solution* $u \in \mathcal{H}^{\gamma+2}_p(T)$*, and*

$$\|u\|_{\mathcal{H}^{\gamma+2}_p(T)} \leq c\left(\|f\|_{\mathbb{H}^\gamma_p(T)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon}_p(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p}\right), \tag{2.16}$$

where $c = c(\delta, K, L, p, \gamma, T)$*.*

Remark 2.10 (i) Note that Theorem 2.9 requires stronger conditions than those in Theorem 2.7; σ^{ik} is assumed to be zero, and the regularity condition of μ is stronger since $|\mu|_{B^{|\gamma+1|_+}} \leq |\mu|_{B^{|\gamma+1|^+}}$.

(ii) However Theorem 2.9 gives better regularity results of solutions; let $\gamma + 2 - d/p > 0$ and u be the solution in the above theorems. Then from the embedding $H^{\gamma+2}_p \subset C^{\gamma+2-d/p}$, it follows that

$$\mathbb{E}\left[\int_0^T |u|^p_{C^{\gamma+2-d/p}} ds\right] \leq c\left(\|f\|_{\mathbb{H}^\gamma_p(T)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon}_p(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p}\right).$$

3 SPDEs with constant coefficients

In this section we consider the equation

$$du = \left(a^{ij}u_{x^ix^j} + f\right)\,dt + \left(\sigma^{ik}u_{x^i} + g^k\right)\,dZ^k_t, \tag{3.1}$$

where the coefficients a^{ij}, σ^{ik} are independent of x. Recall that $\delta_0 > 0$ is the constant so that $(a^{ij}) \geq \delta_0 I_{d\times d}$.

Let T_t denote the semigroup associated with the Laplacian Δ on $\mathbb{R}^d$, that is,

$$T_t f(x) = P_t * f(x), \qquad \text{where} \quad P_t(x) = (2\pi t)^{-d/2}e^{-|x|^2/(2t)}.$$

Lemma 3.1 *Let* $p \in [2,\infty)$ *and* $g = (g^1, g^2, \cdots) \in L^p((0,T)\times\mathbb{R}^d, \ell^2)$*. Then*

$$\int_{\mathbb{R}^d}\int_0^T \left(\int_0^t |DT_{t-s}g(s,x)|^2_{\ell^2}\,ds\right)^{p/2} dtdx \leq c(d,p)\int_{\mathbb{R}^d}\int_0^T |g(t,x)|^p_{\ell^2}\,dtdx. \tag{3.2}$$

Proof. See Lemma 4.1 in [7].

□

Lemma 3.2 *Let $p \in (2,\infty)$ and $f \in L^p((0,T)\times\mathbb{R}^d)$. Then for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d}\int_0^T\int_0^t |DT_{t-s}f(s,x)|^p \, dsdt\, dx \le c\int_0^T \|f(t,\cdot)\|^p_{H^\varepsilon_p}\, dt, \tag{3.3}$$

where $c = c(d,p,\varepsilon,T)$.

Proof. Let $q > p$ be chosen so that $1/p = (1-\varepsilon)/2 + \varepsilon/q$, and define an operator $\mathcal{A}$ by

$$\mathcal{A}f(t,s,x) = \begin{cases} DT_{t-s}f & \text{if } s < t, \\ 0 & \text{otherwise.} \end{cases}$$

Then, due to Lemma 3.1 and the inequality $\|T_{t-s}Df\|_q \le \|Df\|_q$, the linear mappings

$$\mathcal{A} : L^2([0,T], L^2(\mathbb{R}^d)) \to L^2([0,T]\times[0,T]\times\mathbb{R}^d)$$

and

$$\mathcal{A} : L^q([0,T], H^1_q) \to L^q([0,T]\times[0,T]\times\mathbb{R}^d)$$

are bounded. Thus the lemma follows from the interpolation theory; see, for instance, [2, Theorem 5.1.2]. □

Remark 3.3 We suspect that (3.3) is not true if $\varepsilon = 0$, and this is one of main reasons why we assumed $\sigma^{ik} = 0$ in Theorem 2.9.

Here are the main results of this section.

Theorem 3.4 *For every $f \in \mathbb{H}^\gamma_2(T)$, $g \in \mathbb{H}^{\gamma+1}_2(T,\ell^2)$, $u_0 \in U^{\gamma+2}_2$ and $T > 0$, equation (3.1) with initial data u_0 has a unique solution $u \in \mathcal{H}^{\gamma+2}_2(T)$, and*

$$\|u_x\|_{\mathbb{H}^{\gamma+1}_2(T)} \le c\left(\|f\|_{\mathbb{H}^\gamma_2(T)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}\right) \tag{3.4}$$

$$\|u\|_{\mathbb{H}^{\gamma+2}_2(T)} \le ce^{cT}\left(\|f\|_{\mathbb{H}^\gamma_2(T)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}\right), \tag{3.5}$$

where $c = c(\delta_0, d, K)$ is independent of T.

Proof. Owing to Remark 2.2, we may assume $\gamma = -1$. Indeed, suppose that the theorem holds when $\gamma = -1$. Then it is enough to notice that $u \in \mathcal{H}^{\gamma+2}_2(T)$ is a solution of the equation if and only if $v := (1-\Delta)^{(\gamma+1)/2}u$ is a solution of the equation with $\bar{f} := (1-\Delta)^{(\gamma+1)/2}f$, $\bar{g} := (1-\Delta)^{(\gamma+1)/2}g$ and $\bar{u}_0 := (1-\Delta)^{(\gamma+1)/2}u_0$ in place of f, g and u_0, respectively, and

$$\begin{aligned} \|u\|_{\mathbb{H}^{\gamma+2}_2(T)} = \|v\|_{\mathbb{H}^1_2(T)} &\le c\left(\|\bar{f}\|_{\mathbb{H}^{-1}_2(T)} + \|\bar{g}\|_{\mathbb{H}^0_2(T,\ell^2)} + \|\bar{u}_0\|_{U^1_2}\right) \\ &= c\left(\|f\|_{\mathbb{H}^\gamma_2(T)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}\right). \end{aligned}$$

Since the coefficients a^{ij} are independent of x, equation (3.1) can be rewritten as

$$du = \left(\frac{\partial}{\partial x^i} \left(a^{ij} u_{x^j} \right) + f \right) dt + \left(\sigma^{ik} u_{x^i} + g^k \right) dZ_t^k .$$

By Remark 2.9 and Theorem 2.10 in [3], this equation has a unique solution $u \in \mathbb{H}_2^1(T)$, and furthermore there is a constant $c > 0$ independent of $T > 0$ so that

$$\|u_x\|_{\mathbb{L}_2(T)} \leq c \left(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{H}_2^0(T,\ell^2)} + \|u_0\|_{U_2^1} \right).$$

$$\|u\|_{\mathbb{H}_2^1(T)} \leq c e^{cT} \left(\|f\|_{\mathbb{H}_2^{-1}(T)} + \|g\|_{\mathbb{H}_2^0(T,\ell^2)} + \|u_0\|_{U_2^1} \right).$$

The theorem is proved. □

Theorem 3.5 *Let $p > 2$, $\sigma^{ik} = 0$ for each i,k, $\varepsilon > 0$ and $T > 0$. For every $f \in \mathbb{H}_p^\gamma(T)$, $g \in \mathbb{H}_p^{\gamma+1+\varepsilon}(T, \ell^2)$ and $u_0 \in U_p^{\gamma+2}$, equation (3.1) with initial data u_0 has a unique solution $u \in \mathcal{H}_p^{\gamma+2}(T)$, and*

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \leq c(\|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1+\varepsilon}(T,\ell^2)} + \|u_0\|_{U_p^{\gamma+2}}), \tag{3.6}$$

where $c = c(\delta_0, d, p, \varepsilon, K, T)$.

Proof. Again by Remark 2.2, we only need to prove the theorem for $\gamma = -1$. Since the uniqueness of solution of equation (3.1) follows from the uniqueness result of deterministic equations, we only need to show that there is a solution $u \in \mathcal{H}_p^1(T)$ and u satisfies (3.6).

Step 1. First we prove the theorem for the stochastic heat equation:

$$du = \Delta u dt + \sum_{k=1}^\infty g^k dZ_t^k, \quad u(0) = 0. \tag{3.7}$$

Using a standard approximation arguments, without loss of generality, we may and do assume that $g^k = 0$ for all $k \geq N_0 + 1$ and that

$$g^k(t,x) = \sum_{i=0}^{m(k)} I_{(\tau_i^k, \tau_{i+1}^k]}(t) g^{ki}(x),$$

where τ_i^k are bounded stopping times and $g^{ki}(x) \in C_c^\infty(\mathbb{R}^d)$. Define

$$v(t,x) := \sum_{k=1}^{N_0} \int_0^t g^k(s,x) dZ_s^k = \sum_{k=1}^{N_0} \sum_{i=1}^{m(k)} g^{ki}(x)(Z_{t\wedge\tau_{i+1}^k}^k - Z_{t\wedge\tau_i^k}^k)$$

and

$$u(t,x) := v(t,x) + \int_0^t T_{t-s} \Delta v_s \, ds.$$

Then $d(u - v) = (\Delta(u - v) + \Delta v)dt = \Delta u dt$. Therefore

$$du = \Delta u dt + dv = \Delta u dt + g^k dZ_t^k.$$

Also by stochastic Fubini theorem, almost surely,

$$\begin{aligned}
u(t,x) &= v(t,x) + \sum_{k=1}^{N_0} \int_0^t \int_0^s T_{t-s} \Delta g^k(r,x) dZ_r^k ds \\
&= v(t,x) - \sum_{k=1}^{N_0} \int_0^t \int_r^t \frac{\partial}{\partial s} T_{t-s} g^k(r,x) ds dZ_r^k \\
&= \sum_{k=1}^{N_0} \int_0^t T_{t-s} g^k dZ_s^k.
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x^i} u(t,x) = \sum_{k=1}^{N_0} \int_0^t D_i T_{t-s} g^k dZ_s^k.$$

Thus by Burkholder-Davis-Gundy's inequality and (2.10), we have

$$\begin{aligned}
\mathbb{E}\left[|u_x(t,x)|^p\right] &\leq c\mathbb{E}\left[\left(\sum_{k=1}^{N_0} \int_0^t \int |DT_{t-s} g^k|^2 |z|^2 N^k(dz, ds)\right)^{p/2}\right] \\
&\leq c(p)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{\infty} |DT_{t-s} \widehat{g}^k|^2 ds\right)^{p/2} + \int_0^t \sum_{k=1}^{\infty} |DT_{t-s} \widehat{g}^k|^p ds\right].
\end{aligned}$$

By Lemma 3.1, Lemma 3.2 and the inequality $\sum_k |a_k|^p \leq (\sum_k |a_n|^2)^{p/2}$,

$$\mathbb{E}\left[\int_0^T \|Du\|^p \, dt\right] \leq c\mathbb{E}\left[\int_0^T \|g\|^p_{H_p^{\varepsilon}(\ell^2)} \, dt\right]. \tag{3.8}$$

Next we prove (3.6). As before,

$$\mathbb{E}\left[|u(t,x)|^p\right] \leq c(p)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{N_0} |T_{t-s} \widehat{g}^k|^2 ds\right)^{p/2} + \int_0^t \sum_{k=1}^{N_0} |T_{t-s} \widehat{g}^k|^p \, ds\right]. \tag{3.9}$$

Since $(\sum_{k=1}^{N_0} |a_n|^2)^{p/2} \leq N(N_0,p) \sum_{k=1}^{N_0} |a_n|^p$ and $\|T_t f\|_p \leq \|f\|_p$, it easily follows that $u \in \mathbb{H}_p^0(T)$, and consequently $u \in \mathcal{H}_p^1(T)$. By Theorem 2.5 and (3.8),

$$\mathbb{E}\left[\sup_{s \leq t} \|u\|^p_{H_p^{-1}}\right] \leq c(d,p)\left(\|\Delta u\|^p_{\mathbb{H}_p^{-1}(t)} + \|g\|^p_{\mathbb{H}_p^{-1}(t,\ell^2)}\right) \leq c\|g\|^p_{\mathbb{H}_p^{\varepsilon}(t,\ell^2)}.$$

This together with (3.8) and the inequality

$$\|u\|_{H_p^1} = \|(1-\Delta)u\|_{H_p^{-1}} \leq \|u\|_{H_p^{-1}} + \|\Delta u\|_{H_p^{-1}} \leq \|u\|_{H_p^{-1}} + \|Du\|_p$$

prove (3.6). Before we move to next step, we emphasize that (3.9) has nothing to do with (3.6), and it is used only to show that $\|u\|_{\mathbb{H}_p^0(T)} < \infty$.

Step 2. General case. Let $v \in \mathcal{H}_p^1(T)$ be the solution of equation (3.7), where the existence of the solution is obtained in Step 1. Also let $\bar{u} \in \mathcal{H}_p^1(T)$ be the solution of the following equation (see [7, Theorem 4.10])

$$d\bar{u} = (a^{ij}\bar{u}_{x^ix^j} + f + a^{ij}v_{x^ix^j} - \Delta v)dt, \quad \bar{u}(0) = u_0.$$

Then by Step 1 and [7, Theorem 4.10],

$$\|v\|_{\mathbb{H}_p^1(T)} \le c\|g\|_{\mathbb{H}_p^\varepsilon(\ell^2)},$$

$$\|\bar{u}\|_{\mathbb{H}_p^1(T)} \le c(\|v_{xx}\|_{\mathbb{H}_p^{-1}(T)} + \|f\|_{\mathbb{H}_p^{-1}(T)} + \|u_0\|_{U_p^1}).$$

Note that $u := \bar{u} + v$ satisfies

$$du = (a^{ij}u_{x^ix^j} + f)dt + g^k dZ_t^k, \quad u(0) = u_0$$

and estimate (3.6) follows. □

4 Proof of Theorem 2.7 and Theorem 2.9

First we prove the following lemmas.

Lemma 4.1 *For $c > 0$, let $Z_t^k(c) := c^{-1}Z_{c^2t}^k$ and ν_c^k be the Lévy measure of $Z_t^k(c)$. Then*

$$\int_{\mathbb{R}} z^2\nu^k(dz) = \int_{\mathbb{R}} z^2\nu_c^k(dz).$$

Proof. Denote $N_c^k(t, A) = \#\{s \le t; \Delta Z_s^k(c) \in A\}$. Then $N_c^k(t, A) = N^k(c^2t, cA)$ and

$$\nu_c^k(A) = \mathbb{E}\left[N^k(c^2, cA)\right] = c^2\nu^k(cA).$$

Hence the lemma follows from a change of variables. Indeed, let $f(z) = cz$ and $h(z) = z^2/c^2$. Then $\nu_c^k \circ f^{-1} = c^2\nu^k$, and so

$$\int z^2\nu_c^k(dz) = \int h(f(z))\nu_c^k(dz) = \int h(z)c^2\nu^k(dz) = \int z^2\nu^k(dz).$$

□

Consider equation (3.1) with $Z_t^k(c)$ in place of Z_t^k. It follows from (2.4) with $p = 2$ and Lemma 4.1 that Theorem 3.4 and (3.5) holds for this new equation with the same constant C. Recall the definition of B^r from (2.13) and that $r^+ := r + \kappa_0$.

Lemma 4.2 *For $d \ge 1$, $p \ge 2$ and $\gamma \in \mathbb{R}$, there is a constant $N = N(d, p, \gamma) > 0$ so that for every $a \in B^{|\gamma|_+}$ and $u \in H_p^\gamma$,*

$$\|au\|_{H_p^\gamma} \le N|a|_{B^{|\gamma|_+}}\|u\|_{H_p^\gamma}.$$

The same is true for ℓ^2-valued functions a in $B^{|\gamma|_+}$.

Proof. See Lemma 5.2 in [7]. □

Lemma 4.3 *Let $b^i = c = \mu^k = 0$ and suppose that there is a constant $L > 0$ so that*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + +|\sigma^i(t,\cdot)|_{B^{|\gamma+1|_+}} \leq L.$$

Define

$$\beta := \sup_{\omega,x,y} \left(|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^{ik}(t,x) - \sigma^{ik}(t,y)|_{\ell^2} \right).$$

Then there exists $\beta_0 = \beta_0(d,\delta,K) > 0$, **independent of** *$L$, so that if $\beta \leq \beta_0$ then for any solution of $u \in \mathcal{H}_2^{\gamma+2}(T)$ of equation (2.2) we have*

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} \leq ce^{cT} \left(\|f\|_{\mathbb{H}_2^{\gamma}(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}} \right), \tag{4.1}$$

where $c = c(d,\delta,K,L)$.

Proof. Let $u \in \mathcal{H}_2^{\gamma+2}(T)$ be a solution to equation (2.2). Denote

$$a_0^{ij}(t) = a^{ij}(t,0), \quad \sigma_0^{ik}(t) = \sigma^{ik}(t,0),$$

$$f_0 = (a^{ij} - a_0^{ij})u_{x^ix^j} + f, \quad g_0^k = (\sigma^{ik} - \sigma_0^{ik})u_{x^i} + g^k,$$

$$C_0 = \sup_{\omega,t} \left(|a^{ij} - a_0^{ij}|_{B^{|\gamma|_+}} + |\sigma^i - \sigma_0^i|_{B^{|\gamma+1|_+}} \right).$$

Then $du = (a_0^{ij}u_{x^ix^j} + f_0)dt + (\sigma_0^{ij}u_{x^i} + g_0^k)dZ_t^k$. So by Theorem 3.4,

$$\|u_x\|_{\mathbb{H}_2^{\gamma+1}(T)} \leq c(d,\delta,K) \left(\|f_0\|_{\mathbb{H}_2^{\gamma}(T)} + \|g_0\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}} \right).$$

By Lemma 4.2

$$\|(a^{ij} - a_0^{ij})u_{x^ix^j}\|_{H_2^\gamma} \leq N(d,\gamma)|a^{ij} - a_0^{ij}|_{B^{|\gamma|_+}} \|u_{x^ix^j}\|_{H_2^\gamma} \leq NC_0\|u_{xx}\|_{H_2^\gamma} \leq NC_0\|u_x\|_{H_2^{\gamma+1}},$$

and similarly

$$\|(\sigma^i - \sigma_0^i)u_{x^i}\|_{H_2^{\gamma+1}(\ell^2)} \leq NC_0\|u_x\|_{H_2^{\gamma+1}}.$$

Thus the lemma follows if $cNC_0 \leq 1/4$. For $m \geq 1$, denote $a_m^{ij}(t,x) := a^{ij}(t/m^2, x/m)$ and $\sigma_m^{ik}(t,x) := \sigma^{ik}(t/m^2, x/m)$. Then we have

$$|a_m^{ij}(t,\cdot) - a_m^{ij}(t,0)|_{B^{|\gamma|_+}} \leq \beta + m^{-(|\gamma|_+ \wedge 1)}C_0,$$

and we can drop the second term on the right if $\gamma = 0$. Also we have a similar inequality for σ_m^{ik}. Observe that $u_m(t,x) := u(t/m^2, x/m)$ satisfies

$$du_m = (a_m^{ij}(u_m)_{x^ix^j} + f_m)\,dt + (\sigma_m^{ik}(u_m)_{x^i} + g_m^k)\,dZ_t^k(m^{-1}),$$

where $f_m(t,x) := m^{-2}f(t/m^2, x/m)$ and $g^{ik}_m(t,x) := m^{-1}g^{ik}(t/m^2, x/m)$. Then it follows from the above calculations and Lemma 4.1 that for β sufficiently small and m sufficiently large,

$$\|u_{mx}\|_{\mathbb{H}^{\gamma+1}_2(mt)} \leq c\left(\|f_m\|_{\mathbb{H}^{\gamma}_2(mt)} + \|g_m\|_{\mathbb{H}^{\gamma+1}_2(mt,\ell_2)} + \|u_0(\cdot/m)\|_{U^{\gamma+2}_2}\right)$$

for each $t \leq T$. Also, since $\|\cdot\|_{H^\gamma_p}$ norms of $u(t/m^2, x/m)$ and $u(t,x)$ are comparable, one gets inequality (3.4) for each $t \leq T$ in place of T.

By (3.4) and the inequality

$$\|u\|_{H^{\gamma+2}_2} = \|(1-\Delta)u\|_{H^\gamma_2} \leq \|u\|_{H^\gamma_2} + c\|u_x\|_{H^{\gamma+1}_2}$$

it follows that for each $t \leq T$,

$$\|u\|^2_{\mathcal{H}^{\gamma+2}_2(t)} \leq c\left(\|u\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|f\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g\|^2_{\mathbb{H}^{\gamma+1}_2(t)} + \|u_0\|^2_{U^{\gamma+2}_2}\right).$$

This, (2.8) and Gronwall's inequality prove (4.1).

□

Lemma 4.4 *Let $\zeta_n \in C^\infty, n = 1, 2, \cdots$. Assume that for any multi-index α,*

$$\sup_x \sum_n |D^\alpha \zeta_n(x)| \leq M(\alpha), \tag{4.2}$$

where $M(\alpha)$ are some constants. Then there exists a constant $N = N(d, n, \gamma, p, M)$ such that for any $f \in H^\gamma_p$,

$$\sum_n \|\zeta_n f\|^p_{H^\gamma_p} \leq N\|f\|^p_{H^\gamma_p}.$$

If, in addition,

$$\sum_n |\zeta_n(x)|^p > c > 0, \tag{4.3}$$

then

$$\|f\|^p_{H^\gamma_p} \leq N(d, n, \gamma, p, M, c)\sum_n \|\zeta_n f\|^p_{H^\gamma_p}.$$

Proof. See Lemma 6.7 in [7]. □

Proof of Theorem 2.7 Let $\beta_0 > 0$ be the constant in Lemma 4.3. In view of Theorem 3.4 and the method of continuity (see the proof of [3, Theorem 2.11]), we only need to show that a priori estimate (2.15) holds given that a solution $u \in \mathcal{H}^{\gamma+2}_2(T)$ already exists. Take β_0 from Lemma 4.3. Since a^{ij}, σ^i are uniformly continuous we can fix δ_0 so that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell^2} \leq \beta_0/4$$

if $|x-y| \leq 2\delta_0$. Fix a smooth function $\zeta \in C^\infty_c(B_1(0))$ so that $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ if $|x| \leq 1/2$. Take a sequence of smooth functions $\{\zeta_n : n = 1, 2, \cdots\}$ so that $\zeta_n = \zeta(\frac{2(x-x_n)}{\delta_0})$ for some $x_n \in \mathbb{R}^d$, and (4.2) and (4.3) hold.

Then by Lemma 4.4, for each $t \le T$,

$$\|u\|^2_{\mathbb{H}^{\gamma+2}_2(t)} \le N \sum_n \|u\zeta_n\|^2_{\mathbb{H}^{\gamma+2}_2(t)}. \tag{4.4}$$

Denote $\xi_n(x) = \zeta(\frac{x-x_n}{\delta_0})$ and

$$a^{ij}_n(t,x) = \xi_n(x)a^{ij}(t,x) + (1-\xi_n(x))a^{ij}(t,x_n), \quad \sigma^{ik}_n(t,x) = \xi_n(x)\sigma^{ik}(t,x) + (1-\xi_n(x))\sigma^{ik}(t,x_n).$$

Then a_n and σ_n satisfy (2.14) with the same constants δ, K,

$$|a^{ij}_n(t,x) - a^{ij}_n(t,y)| + |\sigma^i_n(t,x) - \sigma^i_n(t,y)|_{\ell^2} \le \beta_0, \quad \forall \omega, t, x, y$$

and $u\zeta_n$ satisfies

$$d(u\zeta_n) = (a^{ij}_n(u\zeta_n)_{x^ix^j} + f_n)dt + (\sigma^{ik}_n(u\zeta_n)_{x^i} + g^k_n)dZ^k_t,$$

where

$$f_n = -2a^{ij}u_{x^i}\zeta_{nx^j} - a^{ij}u\zeta_{nx^ix^j} + b^iu_{x^i}\zeta_n + cu\zeta_n + f\zeta_n$$
$$g^k_n = -\sigma^{ik}u\zeta_{nx^i} + \mu^ku\zeta_n + g^k\zeta_n.$$

By Lemma 4.3 and Lemma 4.2

$$\begin{aligned}
&\|u\zeta_n\|^2_{\mathbb{H}^{\gamma+2}_2(t)} \le c\left(\|f_n\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g_n\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\zeta_n\|^2_{U^{\gamma+2}_2}\right) \\
\le\ & c\left(\|u_x\zeta_{nx}\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|u\zeta_{nxx}\|^2_{\mathbb{H}^{\gamma}_2(t)} + c\|u\zeta_{nx}\|^2_{\mathbb{H}^{\gamma+1}_2(t)} + \|f\zeta_n\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g\zeta_n\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\zeta_n\|^2_{U^{\gamma+2}_2}\right).
\end{aligned}$$

Thus by (4.4) and Lemma 4.4

$$\|u\|^2_{\mathbb{H}^{\gamma+2}_2(t)} \le c\left(\|u\|^2_{\mathbb{H}^{\gamma+1}_2(t)} + \|f\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|^2_{U^{\gamma+2}_2}\right). \tag{4.5}$$

By definition of the space $\mathcal{H}^{\gamma+2}_2(t)$ and Lemma 4.2

$$\begin{aligned}
\|u\|_{\mathcal{H}^{\gamma+2}_2(t)} &:= \|u\|_{\mathbb{H}^{\gamma+2}_2(t)} + \|a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f\|_{\mathbb{H}^{\gamma}_2(t)} + \|\sigma^iu_{x^i} + \mu u + g\|_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2} \\
&\le N\|u\|_{\mathbb{H}^{\gamma+2}_2(t)} + \|f\|_{\mathbb{H}^{\gamma}_2(t)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}.
\end{aligned}$$

This together with (4.5), the embedding inequality (see, for instance, [14])

$$\|u\|^2_{H^{\gamma+1+\beta}_2} \le \varepsilon\|u\|^2_{H^{\gamma+2}_2} + c(\varepsilon,\beta)\|u\|^2_{H^{\gamma}_2}, \quad \forall \beta \in [0,1) \tag{4.6}$$

and Theorem 2.5 yields that

$$\|u\|^2_{\mathcal{H}^{\gamma+2}_2(t)} \le c\int_0^t \|u\|^2_{\mathcal{H}^{\gamma+2}_2(s)}ds + c\left(\|f\|^2_{\mathbb{H}^{\gamma}_2(T)} + \|g\|^2_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|^2_{U^{\gamma+2}_2}\right).$$

The a priori estimate now follows from Gronwall's inequality. The theorem is proved. □

Proof of Theorem 2.9 Again, in view of Theorem 3.5 and the method of continuity, we only need to prove that a priori estimate (2.16) holds given that a solution $u \in \mathcal{H}^{\gamma+2}_p(T)$ already exists.

Without loss of generality we assume that $\varepsilon < (\kappa_0 \wedge 1)$ since once (2.16) holds for some small ε then it holds for any $\varepsilon' \geq \varepsilon$. One can prove the theorem by modifying the proof of Theorem 2.7. But since σ^{ik} is assumed to be zero, the proof of this theorem is much easier.

Let $v \in \mathcal{H}_p^{\gamma+2}(T)$ be the solution of

$$dv = \Delta v dt + (\mu^k u + g^k) dZ_t^k, \quad v(0) = u_0.$$

The existence of the solution of the above equation is guaranteed by Theorem 3.5. By Theorem 3.5 and Lemma 4.2,

$$\|v\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq c\left(\|\mu u + g\|_{\mathbb{H}_p^{\gamma+1+\varepsilon}(t,\ell^2)} + \|u_0\|_{U_p^{\gamma+2}}\right) \leq c\left(\|u\|_{\mathbb{H}_p^{\gamma+1+\varepsilon}(t)} + \|g\|_{\mathbb{H}_p^{\gamma+1+\varepsilon}} + \|u_0\|_{U_p^{\gamma+2}}\right). \tag{4.7}$$

Note that $\bar{u} := u - v \in \mathcal{H}_p^{\gamma+2}(T)$ satisfies

$$d\bar{u} = (a^{ij}\bar{u}_{x^i x^j} + b^i \bar{u}_{x^i} + c\bar{u} + \bar{f})dt, \quad \bar{u}(0) = 0, \tag{4.8}$$

where $\bar{f} = a^{ij} v_{x^i x^j} - \Delta v + b^i v_{x^i} + cv + f$. By Theorem 5.2 in [7],

$$\|\bar{u}\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq c\|\bar{f}\|_{\mathbb{H}_p^{\gamma}(t)} \leq c\left(\|v\|_{\mathbb{H}_p^{\gamma+2}(t)} + \|f\|_{\mathbb{H}_p^{\gamma}(t)}\right).$$

Consequently, for each $t \leq T$, by (4.7) and (4.8)

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq c\left(\|\bar{u}\|_{\mathbb{H}_p^{\gamma+2}(t)} + \|v\|_{\mathbb{H}_p^{\gamma+2}(t)}\right) \leq c\left(\|u\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon}(t)} + \|f\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon}(t)} + \|u_0\|^p_{U_p^{\gamma+2}}\right).$$

This together with the embedding inequality (see (4.6))

$$\|u\|_{H_p^{\gamma+1+\varepsilon}} \leq \delta\|u\|_{H_p^{\gamma+2}} + c(\delta,\varepsilon)\|u\|_{H_p^{\gamma}},$$

yields that

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq c\left(\|u\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|f\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon}(t)} + \|u_0\|^p_{U_p^{\gamma+2}}\right).$$

As in the proof of Theorem 2.7, this easily leads to (2.16). The theorem is proved. □

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